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Two-temperature nonequilibrium Ising models: Critical behavior and universality

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We study a class of two-dimensional (2D) nonequilibrium Ising models based on competing dynamics induced by contact with heat baths at two different temperatures. We make a comparative study of the nonequilibrium versions of Metropolis, heat bath–Glauber, and Swendsen-Wang dynamics, and focus on their critical behavior in order to understand their universality classes. We present strong evidence that some of these dynamics have the same critical exponents, and belong to the same universality class as the equilibrium 2D Ising model. We show that the bond version of the Swendsen-Wang update algorithm can be mapped into an equilibrium model at an effective temperature.

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I. INTRODUCTION

Despite attempts at constructing a rigorous theory for *nonequilibrium* statistical mechanics, there is still no formalism to parallel the one which exists for *equilibrium* systems. As a result, there are few analytical methods with which to deal with nonequilibrium systems. In general, nonequilibrium systems display rich and complex behavior such as phase separation, pattern formation, and turbulence [1,2] and it is therefore useful to first study simple model systems.

There exist model nonequilibrium systems described by stationary distributions that are comparatively easy to study. An open system maintained in a nonequilibrium steady state by an external temperature or density gradient is one example [3]. Another class of nonequilibrium steady states is obtained when the system is closed, and the dynamics is a local competition of two dynamics at different temperatures [14]. It is this type of system that we address in this paper. Since these nonequilibrium systems display behavior qualitatively similar to equilibrium systems, such as phase transitions, it is important to ask if their critical properties and universality classes are the same. Flows of renormalized probability distributions and fixed points are normally independent of the details of a Hamiltonian. Perhaps these flows and fixed points are even independent of the existence of a Hamiltonian and a Boltzmannian distribution. If one were able to establish equivalences (or near equivalences) of universality classes between equilibrium and nonequilibrium models, then we would be able to answer many questions about the behavior of these systems without a complete formulation equivalent to the one for equilibrium systems.

Simple nonequilibrium spin-flip stochastic systems have received considerable attention in the literature over the last ten years. Reviews and general discussions about nonequilibrium phase transitions and stationary states can be found in Refs. [4–7]. Studies of driven diffusive systems can be found in Wang, Binder, and Lebowitz [8], Marro, Garrido, and Vallés [9], Garrido, Marro, and Dickman [10], and Grinstein, Jayaprakash, and Socolar [11].

Grinstein, Jayaprakash, and He [12] studied the statistical mechanics of probabilistic cellular automata using time-dependent Ginzburg-Landau theory. They suggested that any nonequilibrium spin-flip dynamics with up-down symmetry belongs to the same universality class as the equilibrium Ising model. Their argument is based on the observation that under the renormalization group in $d=4-\epsilon$, the dynamical fixed point of the Ising model is stable with respect to all additional analytic terms that preserve the lattice geometry and the spin up-down symmetry.

Kanter and Fisher [13] analyzed the existence of ordered phases in stochastic Ising systems with short-range interactions. They found that the existence of universality for those systems might depend on the details of the interaction and cast doubts on the general applicability of the argument of Grinstein, Jayaprakash, and He [12].

A two-temperature Glauber Ising model was introduced by Garrido, Labarta, and Marro [14] (we will refer

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to this model as GLM) to investigate stationary nonequilibrium states. They obtained a mean-field solution and performed some Monte Carlo simulations on 2D lattices. They found critical behavior qualitatively similar to the equilibrium case.

The nonequilibrium behavior of competing dynamics such as spin flip vs spin exchange has been studied by Garrido, Marro, and Gonzalez-Miranda [15] using hydrodynamic macroscopic equations and Monte Carlo data, and by Wang and Lebowitz [16] using a Monte Carlo renormalization group method. They found evidence of equilibrium Ising behavior and Ising-like exponents.

Tomé, Oliveira, and Santos [17] studied the GLM model for the case when one of the temperatures is negative. They used a dynamical pair approximation to analyze antiferromagnetic steady states and obtained the corresponding phase diagram. Their mean-field renormalization group calculations show evidence in favor of equivalence with the equilibrium Ising universality class.

Blöte *et al.* [18] studied a model similar to GLM in which each sublattice is in equilibrium at a different temperature. They performed Monte Carlo simulations and found strong evidence that the model belongs to the equilibrium Ising universality class.

Marques [19,20] used a mean-field renormalization group calculation to obtain a phase diagram and calculated the exponent ν for the GLM model. Her results compared well with the equilibrium Ising values. Later she extended this technique to two different three state systems, which retain the up-down symmetry [21], to test the conjecture of Grinstein, Jayaprakash, and He [12], and found good agreement with equilibrium Ising exponents.

Recently, de Oliveira [22] analyzed the isotropic majority-vote nonequilibrium model by Monte Carlo and finite size scaling. He found very good agreement for the critical exponents between this model and the equilibrium Ising model and also for Binder's cumulant. In a separate paper, de Oliveira, Mendes, and Santos [23] studied a family of nonequilibrium spin models with up-down symmetry parametrized by a Glauber-like transition rate. They also found good evidence that the critical exponents for this family (except for the limit case of the voter model) are the same as the equilibrium Ising model.

Considerable numerical and analytic evidence has been accumulating in favor of universality and equilibrium Ising exponents for some of these models but the results are not definite yet. Most analytical methods used, such as the mean-field renormalization group method, are of an approximate nature. Monte Carlo simulations have focused mainly on qualitative behavior, and the calculations of critical exponents have not been carried out with high resolution. To understand better and test this equivalence, we have undertaken a detailed, high resolution Monte Carlo study of two-temperature Ising models, and we explore the question of universality using different local and nonlocal update dynamics.

The paper is organized as follows. In Sec. II, we will describe the different two-temperature nonequilibrium models that are the subject of this study. Section III presents detailed results for the Metropolis nonequilibrium dynamics including critical exponents and cumulant behavior. Section IV focuses on the Swendsen-Wang [24] nonequilibrium dynamics. Section V contains a comparative study of all the dynamics. Extension to many temperature models is discussed in Sec. VI and the conclusions are presented in Sec. VII.

II. TWO-TEMPERATURE NONEQUILIBRIUM DYNAMICS

We begin with the two-dimensional Ising model on the square lattice with Hamiltonian

$$H=-J\beta\sum_{\langle ij\rangle}\sigma_i\sigma_j\ , \tag{1}$$

where β is the inverse temperature and J the coupling. A dynamics for the model can be described in terms of a time-dependent probability distribution $P(\sigma,t)$, which evolves according to a master equation,

$$\frac{dP(\sigma,t)}{dt}=\sum_{\sigma'}\{W(\sigma'\leftarrow\sigma)P(\sigma,t)-W(\sigma\leftarrow\sigma')P(\sigma',t)\}\ , \tag{2}$$

where $W(\sigma'\leftarrow\sigma)$ is the transition rate from configuration σ to σ'. We are interested in stationary probability distributions,

$$\frac{dP(\sigma,t)}{dt}=0\ ,\quad P(\sigma,t)=P(\sigma)\ . \tag{3}$$

In the case of equilibrium systems, $P(\sigma)$ is not only stationary but has the form of a Boltzmann distribution parametrized by the inverse temperature β,

$$P(\sigma)=Z^{-1}\exp(-\beta H)\ , \tag{4}$$

where $Z=\sum_\sigma\exp[-\beta H(\sigma)]$. In our case, we are interested in stationary nonequilibrium distributions produced by the local competition of equilibrium dynamics at different temperatures. The usual condition to obtain an equilibrium Monte Carlo dynamics is to make $W(\sigma'\leftarrow\sigma)$ obey detailed balance (note that imposing detailed balance on the W is a sufficient but not necessary condition),

$$W(\sigma'\leftarrow\sigma)P(\sigma)=W(\sigma\leftarrow\sigma')P(\sigma')\ . \tag{5}$$

In the two-temperature model, one considers a composite rate $W(\sigma'\leftarrow\sigma)$,

$$W(\sigma'\leftarrow\sigma)=pW_1(\sigma'\leftarrow\sigma)+(1-p)W_2(\sigma'\leftarrow\sigma)\ , \tag{6}$$

with competing W_1 and W_2. At each time step the transition probability will be chosen at random to be W_1 with probability p, or W_2 with probability $(1-p)$. W_1 and W_2 individually correspond to equilibrium transition rates obeying detailed balance with respect to temperatures β_1 and β_2, i.e.,

$$\frac{W_i(\sigma\leftarrow\sigma')}{W_i(\sigma'\leftarrow\sigma)}=\frac{e^{\beta_i H(\sigma)}}{e^{B_i H(\sigma')}}\ . \tag{7}$$

At each time step the dynamics obeys detailed balance locally with respect to β_1 or β_2, and the spins act as if in instantaneous contact with one of two heat baths. The overall effect is to produce a nonequilibrium dynamics that reduces to the equilibrium model when $\beta_1=\beta_2$. From the master equation one can prove that for a combined dynamics of this sort there is indeed a stationary regime given by the condition [4],

$$p\langle \sigma W_1\rangle+(1-p)\langle \sigma W_2\rangle=0 . \tag{8}$$

The induced global probability distribution $P(\sigma)$ does not, in general, correspond to a local known Hamiltonian, and it depends on the details of the dynamics, i.e., the particular choice of W_1 and W_2. This is in contrast to equilibrium simulations where the Boltzmann distribution is independent of the particular choice of W. The combination of W_1 and W_2 produces a stationary state analogous to a system being driven by an external potential. The ensemble of stationary configurations exhibits physical properties qualitatively similar to equilibrium (i.e., ordering, cluster formation, phase transitions, etc.). This is therefore one of the simplest ways to generate nonequilibrium models from equilibrium ones. In order to investigate the properties of stationary distribution on the dynamics, we study three different update algorithms: Metropolis, Glauber (or equivalently heat bath) and Swendsen-Wang. Furthermore, each of these dynamics has a "bond" or "spin" version, as we explain below.

We start by defining the GLM dynamics [14]. The transition rate for spin i takes the standard Glauber form,

$$W_i=\tfrac{1}{2}\alpha\left[1-\tanh\left[J\beta_i\sigma_i\sum_{|j-i|=1}\sigma_j\right]\right] , \tag{9}$$

where β_i is chosen to be equal to β_1 with probability p, and β_2 with probability $(1-p)$. In one dimension, this dynamics is always equivalent to an equilibrium model at an effective temperature β_{eff} given by [14],

$$\tanh(2J\beta_{\text{eff}})=p\tanh(2J\beta_1)+(1-p)\tanh(2J\beta_2) . \tag{10}$$

Similarly, bond dynamics is defined by

$$W_i=\tfrac{1}{2}\alpha\left[1-\tanh\left[J\sigma_i\sum_{|j-i|=1}\beta_j\sigma_j\right]\right] , \tag{11}$$

where β_j is β_1 with probability p, or β_2 with probability $(1-p)$, and in this way the temperature is selected independently for each bond. These transition rates with $\alpha=1$ also define the heat-bath algorithm, i.e., the two algorithms are equivalent and we only need to discuss one. Henceforth, we shall refer to this as the Glauber dynamics.

Metropolis nonequilibrium dynamics are defined in a similar way. The relevant acceptance factor for spin i is given by

$$A_i=\exp\left[-2J\sigma_i\beta\sum_j\sigma_j\right] , \tag{12}$$

where, as in the Glauber case, β is either β_1 or β_2. In the bond version, we have

$$A_i=\exp\left[-2J\sigma_i\sum_j\beta_j\sigma_j\right] , \tag{13}$$

where each bond j is chosen independently with temperature β_1 or β_2.

Finally, we define nonequilibrium Swendsen-Wang [24] spin and bond versions. In the bond version, the percolation probability for bond ij is chosen to be

$$\pi_{ij}=1-e^{-2J\beta_{ij}} , \tag{14}$$

where β_{ij} is β_1 with probability p, or β_2 with probability $(1-p)$. The subsequent percolation, cluster finding, and flipping steps are done in the same way as in the original Swendsen-Wang dynamics. In Sec. IV, we show that this dynamics can be mapped to an equilibrium system at an intermediate "effective" temperature β_{eff}. The spin version is defined similarly, except that the four bonds contributing to the update of each red (or black) site are chosen to be at the same temperature β_1 or β_2.

All these algorithms can be implemented very efficiently on a parallel computer such as the CM-5. In this paper, we present detailed results for the Metropolis-spin and Swendsen-Wang–bond cases and make some comparisons with the other dynamics.

III. METROPOLIS NONEQUILIBRIUM DYNAMICS

We performed a careful investigation of the spin version of the Metropolis dynamics. Our update algorithm is parallel, so we simultaneously update all the red (black) sites on the (checkerboard) lattice. For each sublattice site i, we choose temperature β_1 or β_2 independently using a uniformly distributed random number. Then we compute the change of energy with the spin flipped,

$$\Delta E_i=2\sigma_i\sum_j\sigma_j . \tag{15}$$

If $\Delta E_i\le 0$, then the flip is always accepted, otherwise it is accepted with probability $\exp(-\beta_i\Delta E_i)$.

To address the question of the existence of an equivalent equilibrium system, we consider the local transition rate for the combined dynamics,

$$e^{-\beta_{\text{eff}}\Delta E}=pe^{-\beta_1\Delta E}+(1-p)e^{-\beta_2\Delta E} . \tag{16}$$

This equation has to be satisfied for all values of ΔE; however, only the cases for $\Delta E>0$ are temperature dependent. (We thank R. Swendsen for bringing this to our attention.) In one dimension there is only one relevant case ($\Delta E=4$), and the equation is satisfied with

$$\beta_{\text{eff}}=-\tfrac{1}{4}\ln[pe^{-4\beta_1}+(1-p)e^{-4\beta_2}] , \tag{17}$$

as was reported by Garrido, Labarta, and Marro [14]. In two dimensions, one has to satisfy two equations (for $\Delta E=4$ and 8),

$$pe^{-4\beta_1}+(1-p)e^{-4\beta_2}=e^{-4\beta_{\text{eff}}^4} , \tag{18}$$

$$pe^{-8\beta_1}+(1-p)e^{-8\beta_2}=e^{-8\beta_{\text{eff}}^8} . \tag{19}$$

For fixed p, β_1, and β_2, each equation has the solution

$$\beta_{\text{eff}}^4=-\tfrac{1}{4}\ln[pe^{-4\beta_1}+(1-p)e^{-4\beta_2}] , \tag{20}$$

$$\beta_{\text{eff}}^8=-\tfrac{1}{8}\ln[pe^{-8\beta_1}+(1-p)e^{-8\beta_2}] , \tag{21}$$

which is shown in Fig. 1 as a function of β_2 for fixed $\beta_1=0.35$ and $p=0.5$. One can see that only for $\beta_1=\beta_2$ is $\beta^4_{\rm eff}=\beta^8_{\rm eff}$. This shows that for the Metropolis spin algorithm the two-temperature system cannot be described by a *nearest neighbor* effective Hamiltonian. The same is true for a triangular lattice with three (six) nearest neighbors, as there are two (three) independent equations.

Monte Carlo simulations show that the configurations generated over time are qualitatively similar to equilibrium configurations, and the system exhibits an Ising-like phase transition. We have explored the critical behavior of this dynamics on 16^2, 32^2, 64^2, and 128^2 lattices. To locate the critical point, we set $p=0.5$ and $\beta_1=0.35$, and searched for the transition as a function of β_2. Our best estimate for the critical point is $\beta_2=0.6372(5)$. The scaling region around this critical point appears to be rather narrow, and the calculation of critical exponents is therefore quite difficult. We choose this particular set of values for β_1 and β_2, as they are far from the Ising critical point ($\beta_1=\beta_2=0.440\,678$), and hope that any nonequilibrium effects will be manifest. We computed the critical exponents using finite size scaling and Binder's cumulant analysis. We accumulated measurements over 5×10^6 for 16^2, 10^7 steps for 32^2 and 64^2 lattices, and 9×10^6 for 128^2.

The data for magnetization and susceptibility for the four different lattice sizes are given in Table I and plotted in Figs. 2 and 3. These figures show a change in the curvature between $\beta_2=0.6370$ and $\beta_2=0.6372$. On the basis of these data, we estimate that the critical coupling is $\beta_2^*=0.6371(1)$. The slopes give an estimate for the exponents β/ν and γ/ν, assuming that the corrections to the leading finite size scaling forms $m\sim L^{-\beta/\nu}$ and $\chi\sim L^{\gamma/\nu}$ are negligible. The results are

$$\beta/\nu=0.122(2)\ , \tag{22}$$

$$\gamma/\nu=1.73(2)\ , \tag{23}$$

TABLE I. Data for susceptibility, magnetization, and Binder's cumulant for the two-temperature Metropolis spin dynamics ($\beta_1=0.35, p=0.5$).

β_2	L	χ	$\|m\|$	U
0.6372	16	9.11(5)	0.7043(3)	0.611(1)
0.6300	32	33.5(3)	0.629(2)	0.603(3)
0.6350	32	32.0(4)	0.639(2)	0.608(4)
0.6370	32	30.2(8)	0.647(3)	0.611(11)
0.6372	32	30.6(3)	0.6464(7)	0.611(2)
0.6375	32	30.6(3)	0.6463(8)	0.611(3)
0.6400	32	30.0(6)	0.651(2)	0.613(7)
0.6475	32	27.2(4)	0.666(1)	0.619(3)
0.6500	32	26.3(6)	0.671(2)	0.621(5)
0.6550	32	25.1(4)	0.679(1)	0.624(4)
0.6600	32	23.3(7)	0.689(2)	0.628(5)
0.6300	64	114(4)	0.566(4)	0.600(13)
0.6350	64	112(5)	0.580(4)	0.605(12)
0.6370	64	103(2)	0.592(2)	0.611(4)
0.6372	64	101(2)	0.594(2)	0.612(6)
0.6375	64	101(2)	0.594(2)	0.612(5)
0.6400	64	96(5)	0.605(2)	0.616(3)
0.6475	64	76(5)	0.635(4)	0.629(14)
0.6500	64	68(3)	0.644(3)	0.633(11)
0.6550	64	64(5)	0.655(3)	0.636(11)
0.6600	64	56(3)	0.669(2)	0.640(4)
0.6350	128	409(51)	0.52(1)	0.597(40)
0.6370	128	357(22)	0.538(6)	0.607(22)
0.6372	128	315(12)	0.550(2)	0.615(7)
0.6375	128	328(18)	0.550(3)	0.615(11)
0.6400	128	257(20)	0.573(5)	0.627(21)
0.6475	128	197(19)	0.611(3)	0.639(9)
0.6500	128	148(10)	0.626(3)	0.645(10)
0.6550	128	133(15)	0.644(3)	0.649(9)
0.6600	128	91(10)	0.664(2)	0.654(7)

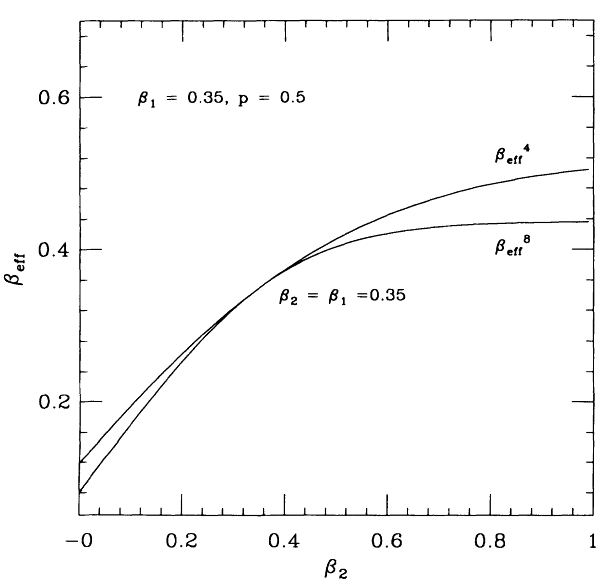


FIG. 1. Plot of $\beta^4_{\rm eff}$ and $\beta^8_{\rm eff}$ [see Eq. (20)] vs β_2 (at $\beta_1=0.35$ and $p=0.5$) for the nonequilibrium Metropolis spin dynamics.

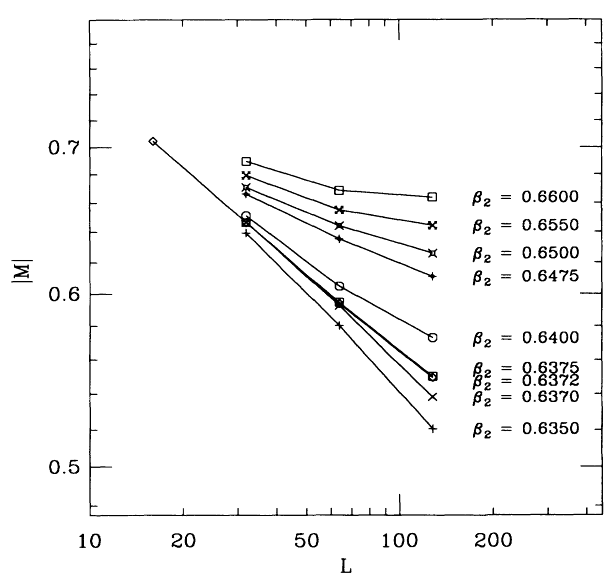


FIG. 2. Scaling behavior of magnetization as a function of β_2 (with $\beta_1=0.35$ and $p=0.5$ held fixed) on different lattice sizes for the nonequilibrium Metropolis spin dynamics.

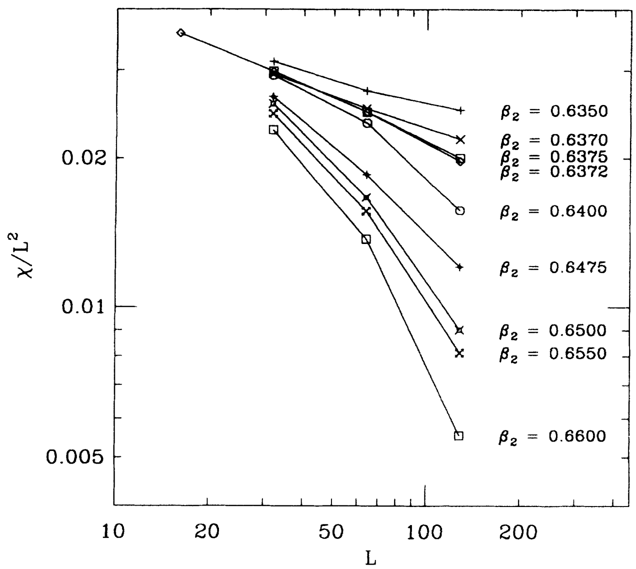


FIG. 3. Scaling behavior of susceptibility as a function of β_2 (with $\beta_1=0.35$ and $p=0.5$ held fixed) on different lattice sizes for the nonequilibrium Metropolis spin dynamics.

where the errors are determined as follows. We first compute the statistical error in the magnetization and susceptibility for each data point. The error in the exponents is then obtained from the mean-square fit to a straight line on a log-log plot.

The data for Binder's cumulant [25],

$$U=1-\frac{\langle M^4\rangle}{3\langle M^2\rangle^2}, \tag{24}$$

are also given in Table I and plotted in Fig. 4 as a function of β_2. The value of this cumulant at the critical point is conjectured to be a universal number independent of lattice size [25]. Figures 5(a) and 5(b) show the cumulant values for the two pairs of lattice sizes, (32^2, 64^2) and (64^2, 128^2), at different temperatures around the critical point. The solid straight line is the best fit to the data and the dashed line in the figures corresponds to $U_{L/2}=U_L$. The point of crossing of these two lines gives an estimate of the critical point. The crossing takes place at

$$U^*(32 \text{ vs } 64)=0.610(2), \tag{25}$$

$$U^*(64 \text{ vs } 128)=0.605(10), \tag{26}$$

and the corresponding estimates for β_2^* are 0.6370(2) and 0.6360(12), respectively. These values are consistent with the estimate from finite size scaling given above. Our longest runs were done at $\beta_2=0.6372$, at which tempera-

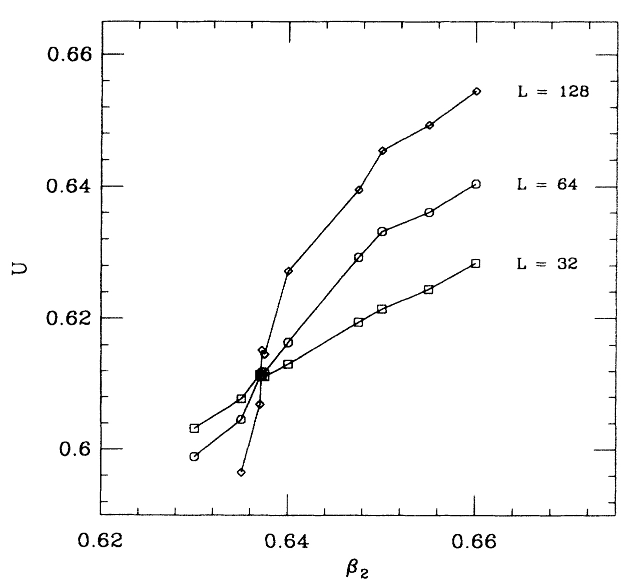


FIG. 4. Cumulant values as a function of β_2 (at $\beta_1=0.35$ and $p=0.5$) and on different lattice sizes for the nonequilibrium Metropolis spin dynamics.

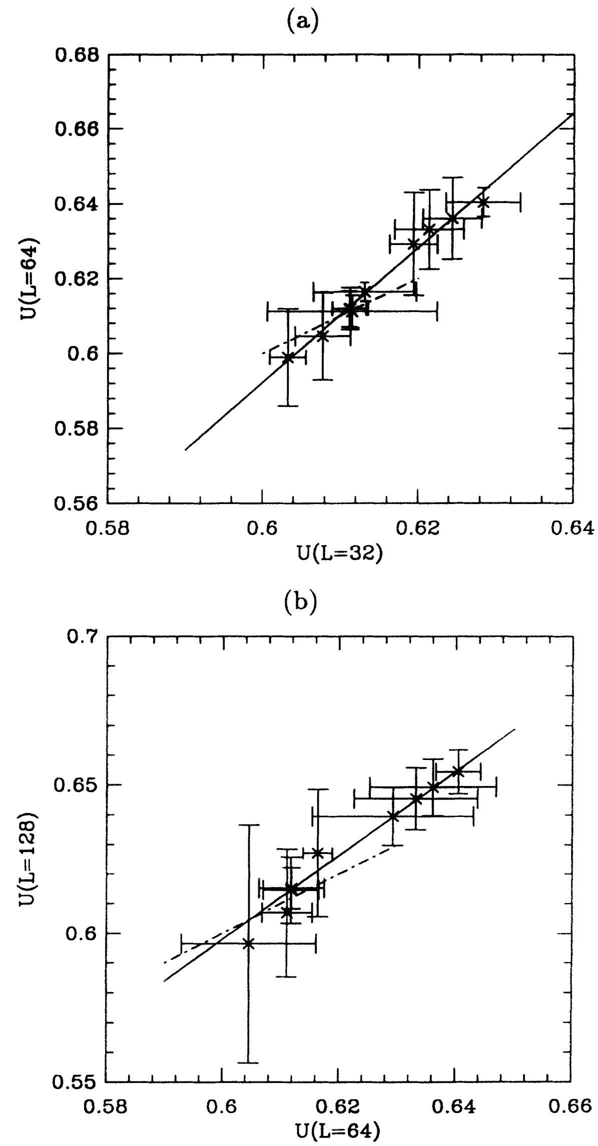


FIG. 5. Cumulant values at temperatures around the critical point for the nonequilibrium Metropolis spin dynamics. (a) 32^2 vs 64^2 lattices; (b) 64^2 vs 128^2 lattices.

ture our estimates,

$$U_{16}=0.611(1)\ , \tag{27}$$

$$U_{32}=0.611(2)\ , \tag{28}$$

$$U_{64}=0.612(6)\ , \tag{29}$$

$$U_{128}=0.615(7)\ , \tag{30}$$

compare very well with the value $U^*=0.611(1)$ computed by Bruce [26] for the 2D equilibrium Ising model, and $U_{64}^*=0.611(5)$ computed independently by us. Thus, we shall henceforth call $\beta_2=0.6372$ the critical point.

From the scaling of the cumulant one can obtain an estimate of ν,

$$\left.\frac{dU_L}{d\beta}\right|_{\beta_c}=L^{1/\nu}G(L^{1/\nu}\epsilon)\ , \tag{31}$$

where $\epsilon=(\beta_c-\beta)/\beta$. To do this, we first compute the slope $dU_L/d\beta$ for $L=32$, 64, and 128 using the data points near the critical point and then fit these values versus L using the above expression to obtain an estimate for ν. We find

$$\nu=0.99(5)\ . \tag{32}$$

Another estimate can be computed from $dU_L/dU_{L'}$ in the critical region because this quantity should scale as

$$\frac{dU_L}{dU_{L'}}\sim\left[\frac{L}{L'}\right]^{1/\nu}\ . \tag{33}$$

Using the more precise data shown in Fig. 5(a) for $L=32$ and 64, we obtain $\nu=0.95(8)$. The agreement between these results and the values for the equilibrium Ising model is quite good [$\nu=1$, $\beta/\nu=0.125$, $\gamma/\nu=1.75$, $U^*=0.611(1)$] and provides strong evidence for their equivalence.

To confirm further this equivalence, we measured the probability distribution for the magnetization and energy at $\beta_1=0.35$ and $\beta_2=0.6372$ and compared them with those for the equilibrium model in Figs. 6(a) and 6(b). The agreement is somewhat better for the magnetization than for the energy, and qualitatively the distribution functions are equilibriumlike. The data are slightly more disordered than those for the critical equilibrium model, suggesting that β_2^* may be slightly larger than 0.6372. The bond version of the Metropolis dynamics has not been studied as extensively, and we postpone its discussion until Sec. V, where we compare the different dynamics [27].

IV. SWENDSEN-WANG NONEQUILIBRIUM DYNAMICS

We start with the bond contribution to a global probability distribution,

$$P_{ij}=pe^{-\beta_1(1-\sigma_i\sigma_j)}+(1-p)e^{-\beta_2(1-\sigma_i\sigma_j)}\ . \tag{34}$$

If we assume that P_{ij} is always equivalent to an equilibrium distribution with coupling $\beta_{\rm eff}$, i.e.,

$$P_{ij}=pe^{-\beta_1(1-\sigma_i\sigma_j)}+(1-p)e^{-\beta_2(1-\sigma_i\sigma_j)}=e^{-\beta_{\rm eff}(1-\sigma_i\sigma_j)}\ , \tag{35}$$

then there exists a solution satisfying this equation for the two possible values of the bond energy,

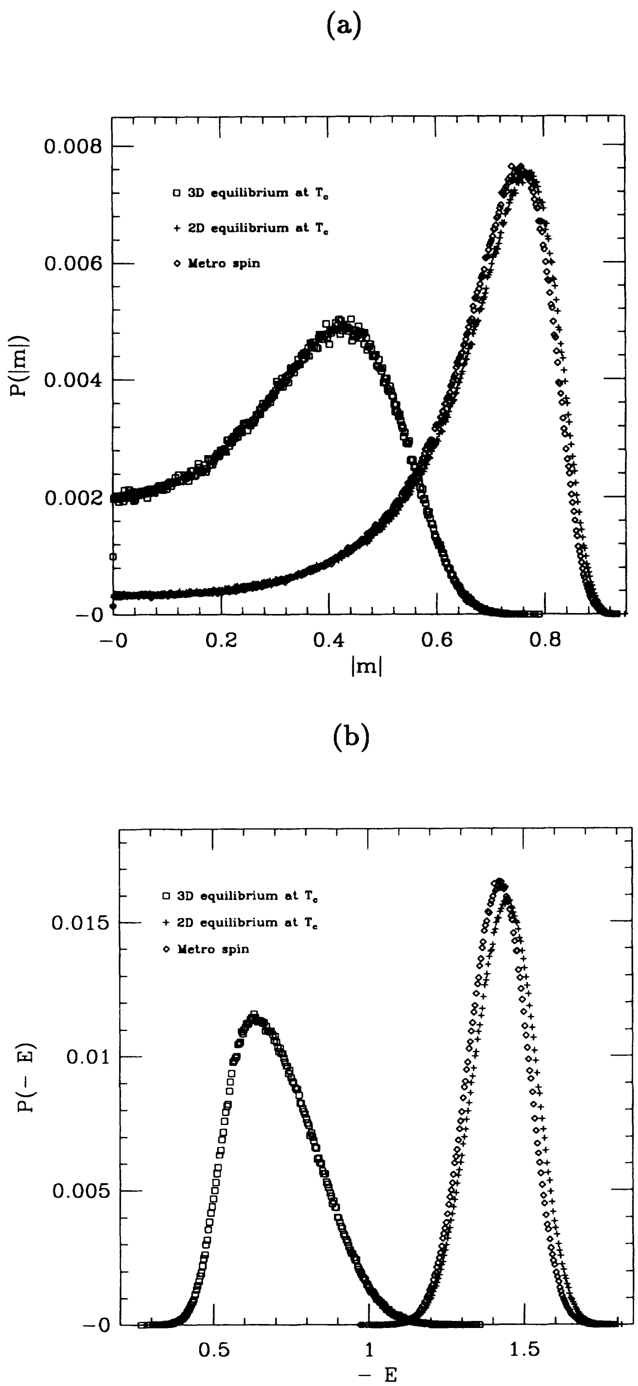


FIG. 6. Probability distributions: (a) magnetization $P(|m|)$ and (b) energy $P(-E)$, for the Metropolis spin dynamics at $\beta_1=0.35$, $\beta_2=0.6372$, $p=0.5$, and $L=32$ (critical point). The probability distributions for the 2D and 3D equilibrium Ising model are also shown for comparison.

$$pe^{-2\beta_1}+(1-p)e^{-2\beta_2}=e^{-2\beta_{\text{eff}}}\ ,\quad [\sigma_i=-\sigma_j]\ , \tag{36}$$

$$p+(1-p)=1\ ,\quad [\sigma_i=\sigma_j]\ . \tag{37}$$

The solution is

$$p=\frac{e^{-2\beta_{\text{eff}}}-e^{-2\beta_2}}{e^{-2\beta_1}-e^{-2\beta_2}}\ , \tag{38}$$

or equivalently, for a set of values β_1, β_2, and p, there always exists a β_{eff} given by

$$\beta_{\text{eff}}=-\tfrac{1}{2}\ln[p(e^{-2\beta_1}-e^{-2\beta_2})+e^{-2\beta_2}] \tag{39}$$

that corresponds to an Ising equilibrium system. This implies that this dynamics is nothing more than equilibrium dynamics in disguise. If we set $\beta_{\text{eff}}=\beta_c$, we find lines of critical points in the $\beta_1-\beta_2$ plane given by

$$\beta_1=-\tfrac{1}{2}\ln\left[\frac{1}{p}[e^{-2\beta_c}-(1-p)e^{-2\beta_2}]\right]\ . \tag{40}$$

A set of these critical lines is shown in Fig. 7 for different values of p. Our simulations confirm this equivalence. Notice that for some extreme values of β_1 (β_2) there is no positive value of β_2 (β_1) that yields critical behavior.

The reason for the equivalence is that for the Swendsen-Wang bond dynamics there are, independent of the number of spatial dimensions, only two equations of constraint, as each bond is independently in contact with the heat bath. One is the trivial condition $p+(1-p)=1$ and the second is the desired result given in Eq. (39). In a different context, a similar analysis of different ways to satisfy local equations for block percolation in equilibrium systems has been made in Ref. [28].

Lastly, we have measured autocorrelation times at criticality with $\beta_1=0.1$, $\beta_2=2.318$, and $p=0.5$ ($\beta_{\text{eff}}=0.440\,687$) to see if the stochastic choice of temperatures accelerates the decorrelation process. We find that the autocorrelation times are comparable to the values for the standard equilibrium Swendsen-Wang values. This is as expected since the two models are locally equivalent.

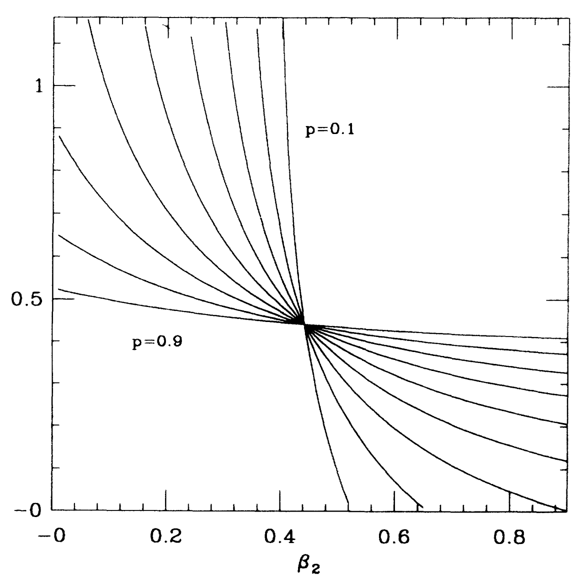


FIG. 7. Critical lines [see Eq. (40)] for the Swendsen-Wang bond dynamics.

V. OTHER DYNAMICS AND COMPARISONS

We have investigated the spin and bond version of Glauber dynamics for only three combinations of β_1 and β_2 using 32^2 lattices. The ensemble size in these runs is $\sim 50\,000$ update sweeps, significantly smaller than for Metropolis or Swendsen-Wang dynamics. In Figs. 8(a), 8(b), and 8(c), we compare the equation of state ($\langle E\rangle$ vs $\langle m^2\rangle$) for the various dynamics for the three different combinations of β_1 and β_2. Figure 8(a) shows the results for $\beta_1=0.4$, $\beta_2=0.6$, Fig. 8(b) corresponds to $\beta_1=0.1$, $\beta_2=2.318$, and Fig. 8(c) to $\beta_1=0.2$, $\beta_2=0.738\,464$. These three sets of temperatures were chosen so that the corresponding β_{eff} for Swendsen-Wang bond dynamics corresponds to a cold ($\beta_{\text{eff}}=0.49$), critical ($\beta_{\text{eff}}=0.440\,686\,8$), and hot ($\beta_{\text{eff}}=0.40$) system, respectively. All the simulations were done with $p=0.5$. The dashed line corresponds to the result for the equilibrium Ising model obtained on $L=32^2$ lattices. The errors in the data are roughly equal to the size of the symbols.

From Fig. 8, it is clear that the equation of state depends very sensitively on the dynamics, and there is a large spread in the results for all three choices of temperatures. In all cases, except for the Swendsen-Wang spin version, the results lie very close to the line characterizing the equilibrium Ising model. The deviations from the equilibrium results in all three versions of the two-temperature spin dynamics are in the direction of a more ordered system. The situation is reversed for the bond dynamics; compared to the equilibrium values the two-temperature results are less ordered. Qualitatively, the data with the various dynamics show the following ordering: Metropolis spin, Glauber spin, Metropolis bond, Glauber bond, and Swendsen-Wang bond. This pattern is also shown in Figs. 9(a) and 9(b), where we give the magnetization and energy probability distributions for $\beta_1=0.35$ and $\beta_2=0.6372$ (a critical point for the Metropolis spin dynamics). All cases appear to have the same functional form, but their position and amplitudes are rescaled. The magnetization probability distributions are Gaussian near the peak but have long tails toward $m=0$. Further work is needed to explore the possibility that all of these probability distributions are just rescaled forms of the equilibrium distribution and correspond to different β_{eff}'s.

VI. THREE-TEMPERATURE MODEL

To study the general case of the competition of many temperatures, we extended the analysis to the three-temperature Metropolis spin dynamics. The three temperatures are chosen with probabilities p_1, p_2, and p_3:

$$W(\sigma'\leftarrow\sigma)=p_1W_1(\sigma'\leftarrow\sigma)+p_2W_2(\sigma'\leftarrow\sigma)+p_3W_3(\sigma'\leftarrow\sigma)\ . \tag{41}$$

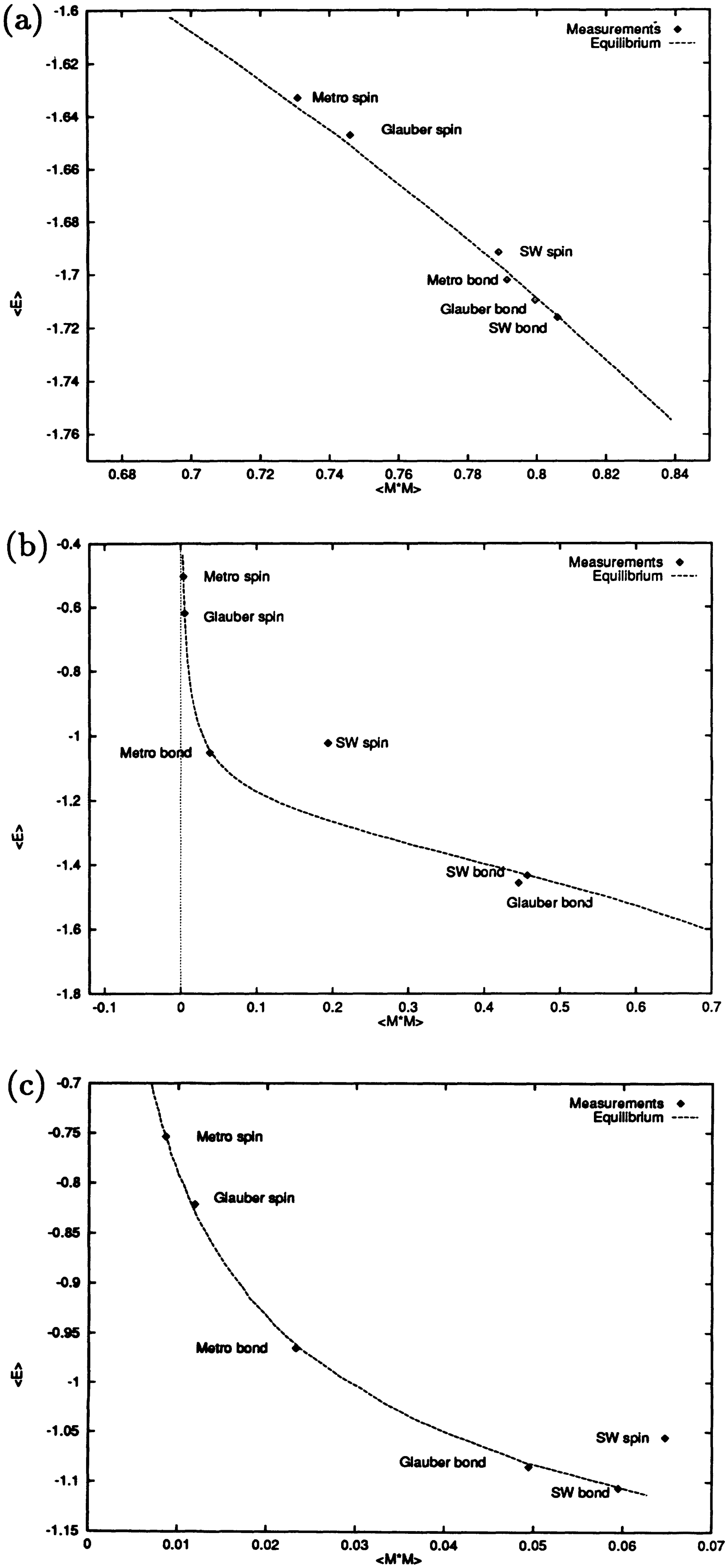
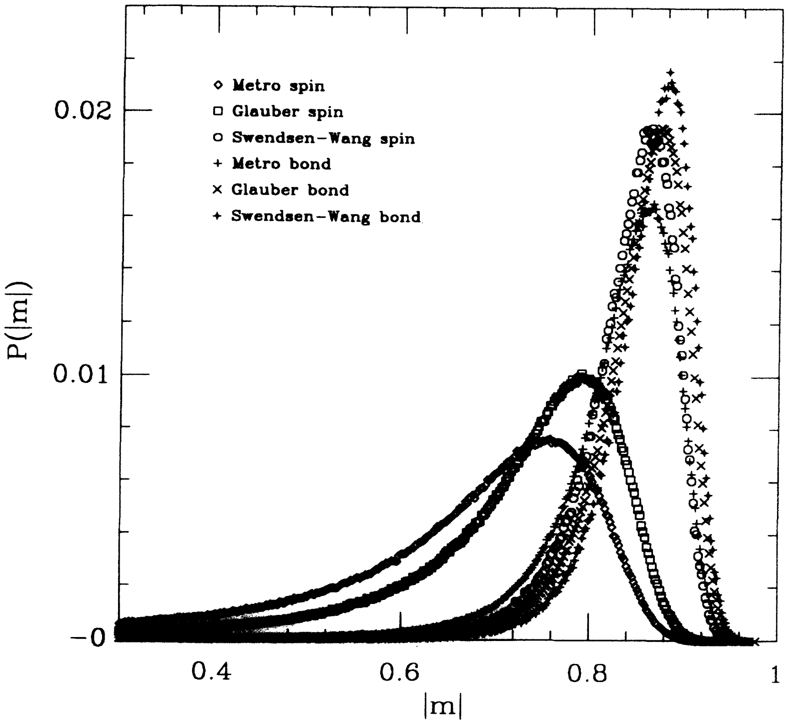


FIG. 8. Equation of state data for different dynamics; (a) $\beta_1=0.4$, $\beta_2=0.6$, $p=0.5$, and $L=32$ ($\beta_{\mathrm{eff}}^{SW\,\mathrm{bond}}=0.49$); (b) $\beta_1=0.1$, $\beta_2=2.318$, $p=0.5$, and $L=32$ ($\beta_{\mathrm{eff}}^{SW\,\mathrm{bond}}=0.440\,686\,8$); and (c) $\beta_1=0.2$, $\beta_2=0.738\,464$, $p=0.5$, and $L=32$ ($\beta_{\mathrm{eff}}^{SW\,\mathrm{bond}}=0.4$).

(a)



(b)

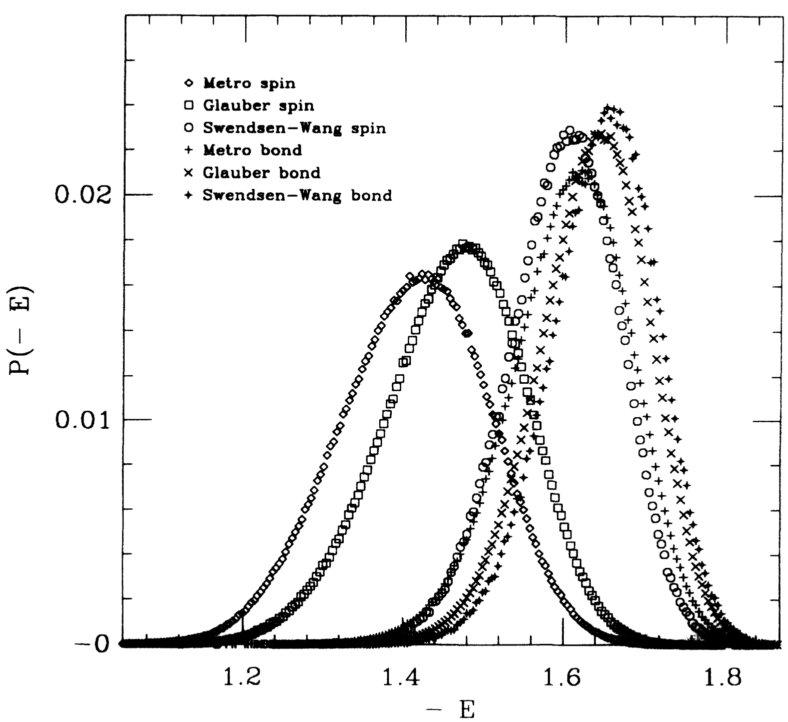


FIG. 9. Probability distributions: (a) magnetization $P(|m|)$ and (b) energy $P(-E)$ for the spin and bond versions of Metropolis, Glauber, and Swendsen-Wang dynamics at $\beta_1=0.35$, $\beta_2=0.6372$, $p=0.5$, and $L=32$ (critical point).

We fixed $\beta_1=0.35$, $\beta_3=0.6372$, and $p_1=p_2=p_3=\frac{1}{3}$ and varied β_2 about the equilibrium critical value $\beta_2=0.440\,686\,8$. We expected the system to display critical behavior for $\beta_2=0.440\,686\,8$.

The system effectively displays critical behavior in that region as can be seen in Fig. 10, where results for Binder's cumulant are shown for β_2 in the range [0.424, 0.456]. Furthermore, the probability distribution functions for $\beta_2=0.440\,686\,8$ match the ones for the two-temperature model discussed in Sec. IV and the equilibrium model at criticality, as can be seen in Figs. 11(a) and 11(b). We do find an apparent narrowing of the critical region compared with the two-temperature models, and the statistical quality of the data are not accurate enough to measure the exponents.

Based on the study of the three-temperature model, we make the following conjecture for the Metropolis spin dynamics. A model in contact with an arbitrary number of heat baths will display Ising critical behavior, provided each temperature or a pair of them is tuned to the critical value. As the number of pairs of temperatures increase, the critical region becomes narrower, making the measurement of critical exponents and the study of critical behavior more difficult.

VII. CONCLUSIONS

We present high statistics results showing that for the Metropolis spin dynamics the stationary states produced by the two-temperature model are very similar to equilibrium states. Based on the agreement of the critical exponents and Binder's cumulant, we conclude that the two-temperature Metropolis spin dynamics is in the same universality class as the Ising model. We also show that the bond version of the two-temperature Swendsen-Wang dynamics can be mapped into an equilibrium Ising model at an intermediate effective temperature. Thus, for these cases our results agree with the conjecture of Grinstein, Jayaprakash, and He that any nonequilibrium spin-flip dynamics that preserves up-down symmetry belongs to the same universality class as the equilibrium Ising model. Assuming that the system evolves into an equilibrium distribution after some thermalization steps, one can measure the critical exponents and flow of renormalized

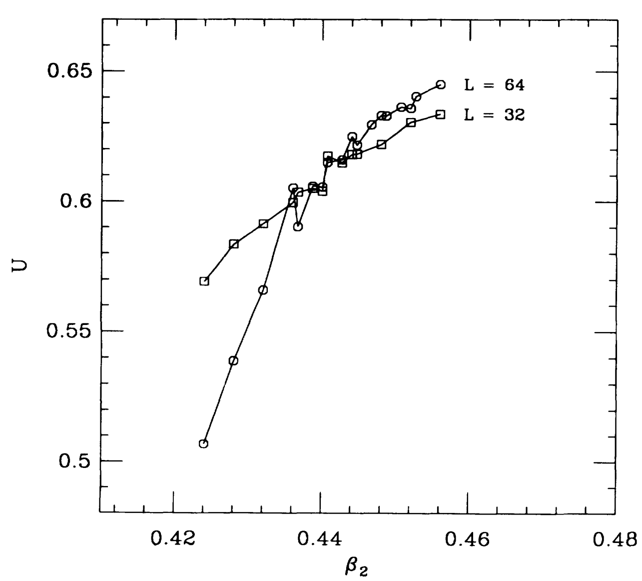


FIG. 10. Cumulant values as a function of β_2 (at $\beta_1=0.35$ and $\beta_3=0.6372$) and $L=32$, and 64 lattice sizes for the three-temperature Metropolis spin dynamics.

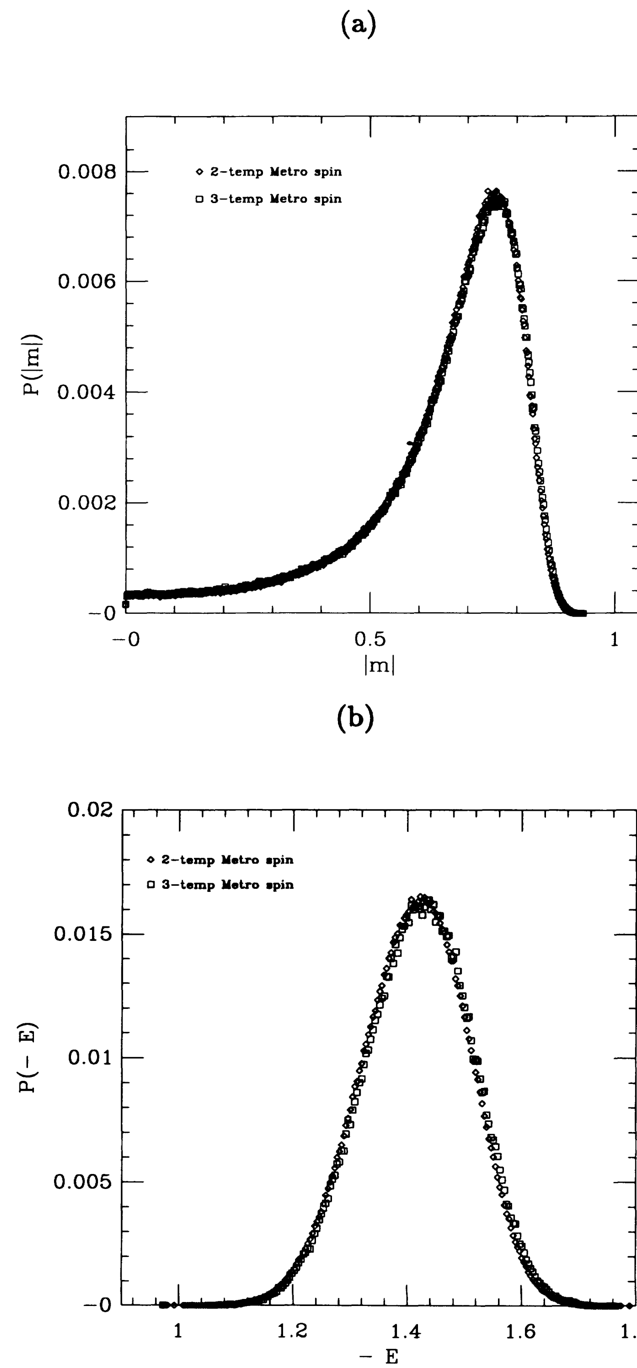


FIG. 11. Probability distributions: (a) magnetization $P(|m|)$ and (b) energy $P(-E)$ for the two-temperature and three-temperature Metropolis spin dynamics at the critical point ($L=32$).

couplings using the Monte Carlo renormalization group method. We hope to investigate this possibility in the future.

Our results for the Metropolis bond, Glauber spin and bond, and Swendsen-Wang spin dynamics are qualitative. Further work is required to confirm that they too belong to the same universality class as the equilibrium Ising model.

We have extended the two-temperature critical Metropolis spin dynamics to the three-temperature case. We find that the system shows critical behavior when the third temperature is tuned to $\beta=0.4406868$. Based on this, we conjecture that the Ising critical behavior is preserved as long as one adds pairs of temperatures that are, by themselves, critical. The critical region appears to become narrower as the number of pairs of temperature values are increased and the statistical quality of the data deteriorates. This makes the calculation of critical exponents and the study of the models more difficult.

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